Calculation of Sonic Boom Signatures by Bicharacteristic Methods

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An improved method is described for correcting the zeroth order (freestream) characteristics to obtain a uniformly valid first order solution to the exact equations of motion for a compressible fluid. The method is based on constructing first order bicharacteristic lines from the system of ordinary differential equations belonging to the exact equation for the characteristic surface. Calculated sonic boom signatures are compared with experiments conducted at Langley Research Center on a cone-cylinder at Mach numbers 2.96, 3.83, and 4.63. Theory and experiment agree well over the entire Mach number range.

Nomenclature

c = sound speed
F = Whitham F-function, Eq. (14)
k = $-(\gamma - 1)u_0$
M = freestream Mach number
P = static pressure
ΔP = static pressure rise relative to freestream
p = axial direction number of bicharacteristic
q = radial direction number of bicharacteristic
$\varepsilon \bar{R}$ = radius of slender body
r = radial coordinate
S = bicharacteristic parameter, Eq. (15)
T = bicharacteristic parameter, Eq. (15)
t = bicharacteristic parameter, measure of arc length along bicharacteristic
u = total axial speed
v = total radial speed
W = shock wave speed
x = axial coordinate
X_0 = distance from vertex to shoulder of cone-cylinder
Z = F-function parameter, measure of arc length along F-function
β = $(M^2 - 1)^{1/2}$
γ = ratio of specific heats ($\gamma = 1.4$)
δ = radius of cylindrical portion of cone-cylinder
ε = small parameter characterizing disturbing body
λ = arbitrary constant
ρ = local density
τ = bicharacteristic parameter, Eq. (6)
$\varphi(x, r)$ = bicharacteristic
$\psi(x, r)$ = 0 shock wave

Subscripts
B = relating to bounding body
0 = freestream conditions, $0(\varepsilon^0)$
1 = first-order quantity, $0(\varepsilon^1)$
2 = second order quantity, $0(\varepsilon^2)$

I. Introduction

RECENT experiments indicate that sonic boom pressure signatures measured at freestream Mach numbers greater than about 2 start to depart from signatures that are calculated from the basic Whitham theory. This discrepancy is most obvious in the slope of the expansion portion of the signature. It is apparent that the theory must somehow be modified to extend its range of validity to higher Mach numbers, but the precise parameters that are most sensitive to Mach number variations are not easy to perceive. Attempts to resolve these discrepancies have attributed the lack of agreement to either higher order nonlinear effects,[1] the use of the approximate F-function,[2] or to near-field effects.[3] All these effects influence the pressure signatures, but it is the intention of this paper to show that the major cause of the discrepancy may be traced to an imprecise location of the first-order characteristics (at least for slender bodies). To isolate this characteristic effect only, the F-function form of the linearized solution will be used throughout. It is quite simple, in principle, to include a more refined linear solution in the analysis.

The Whitham method[4] for calculating a uniformly valid first order representation to the velocity field is based on a procedure whereby only the streamwise variable (say x) is stretched. In the present method the first-order bicharacteristic lines are obtained by stretching all of the independent variables simultaneously.† The parameters with respect to which the stretched independent variables are expressed arise naturally from the initial value problem for the first-order partial differential equation for the characteristics. Any shock waves in the flow are fitted into the field of first-order bicharacteristics by applying the Rankine-Hugoniot shock relations.

To simplify the manipulations as much as possible, the procedure has been applied to a slender body of revolution. In this case rotational symmetry reduces the number of independent variables to two, but it is not difficult to extend the procedure to fully three-dimensional problems. In the case of axisymmetry the bicharacteristics are planar curves, but they are generally space curves for three-dimensional problems. The case of nonuniform upstream flow is also straightforward because the zeroth-order approximation to the bicharacteristics become the ray equations of geometrical acoustics.[5]

The theory has been worked out completely for the case of a slender cone-cylinder. Comparisons with experiment and Whitham theory show that significantly better agreement with experiment occurs when the bicharacteristic method is used at higher Mach numbers. It is concluded that the major discrepancy at moderately high Mach numbers is due to an imprecise location of the first-order characteristics. Even the simple F-function is a good enough approximation to the linear solution for slender configurations.

In Sec. II the bicharacteristics are calculated to second-order accuracy‡ for a slender axisymmetric configuration. Section III is devoted to a description of the calculation procedure for the

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† The bicharacteristics are the characteristics of the characteristic partial differential equation. They are space curves which form the generators of the characteristic surface.

‡ For axisymmetric flows the first nonvanishing deviations of the flow parameters from the undisturbed stream are second-order quantities.

shock wave. The method presented for the shock wave calculation may not be the simplest for two independent variables, but it is most general in that the case of three independent variables can be treated in an analogous manner. In Sec. IV the theory is applied to a slender cone-cylinder. Finally, comparisons of experiment with both the bicharacteristic and conventional theories are presented.

II. Parametric Representation of the Bicharacteristics

The equations of motion for a steady, irrotational, isentropic flow with rotational symmetry are[6]

$$v_x - u_r = 0$$
$$(c^2 - u^2)u_x - uv(u_r + v_x) + (c^2 - v^2)v_r + c^2(v/r) = 0 \quad (1)$$

In Eq. (1) u and v are the velocity components in the x and r directions, c is the local sound speed, and suffix x or r indicates partial differentiation. The sound speed is related to u and v through Bernoulli's Equation for an ideal gas

$$(u^2 + v^2) + [2/(\gamma - 1)]c^2 = \text{const} \quad (2)$$

The first-order system given by Eq. (1) can be expressed in matrix form as

$$[A]\begin{bmatrix} u_x \\ v_x \end{bmatrix} + [B]\begin{bmatrix} u_r \\ v_r \end{bmatrix} + \begin{bmatrix} 0 \\ c^2(v/r) \end{bmatrix} = 0$$

$$[A] = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & -uv \end{bmatrix}$$

$$[B] = \begin{bmatrix} -1 & 0 \\ -uv & c^2 - v^2 \end{bmatrix} \quad (3)$$

Let the characteristic corresponding to Eq. (3) be denoted by $\varphi(x, r) = \text{const}$. The partial differential equation for φ is defined by[7]

$$H = \det\{[A](\partial\varphi/\partial x) + [B](\partial\varphi/\partial r)\}$$

On expanding the determinant,

$$H = (up + vq)^2 - c^2(p^2 + q^2) = 0 \quad (4)$$

where the direction numbers p and q represents $\partial\varphi/\partial x$ and $\partial\varphi/\partial r$. This equation simply represents the condition that the magnitude of the velocity vector normal to φ = constant is equal to the local sound speed c. Equation (4) can be solved by constructing the characteristic lines belonging to this first-order partial differential equation. However, Eq. (4) is itself the characteristic equation belonging to the equations of motion given by Eq. (1); the characteristics of Eq. (4) are therefore the bicharacteristics of Eq. (1).

The bicharacteristic lines are given by the following system of five ordinary differential equations[7]

$$dx/dt = \lambda(\partial H/\partial p), \quad dp/dt = -\lambda(\partial H/\partial x)$$
$$d\varphi/dt = p(\partial H/\partial q) + q(\partial H/\partial q) \quad (5)$$
$$dr/dt = \lambda(\partial H/\partial q), \quad dq/dt = -\lambda(\partial H/\partial r)$$

for the quantities $x(t)$, $r(t)$, $p(t)$, $q(t)$, and $\varphi(t)$. In Eq. (5) t is a parameter and λ is an arbitrary constant. These equations must be supplemented by the boundary conditions. Let the boundary be a slender, axisymmetric body situated near the axis $r = 0$. If a small parameter ε characterizes the lateral dimensions of the body, the surface can be represented parametrically by

$$x_B = \tau$$
$$r_B = \varepsilon \bar{R}(\tau) \quad (6)$$

The value of the characteristic parameter on the body can also be labeled by the parameter τ

$$\varphi_B = \tau \quad (6a)$$

It remains to determine the boundary values for p and q. They are determined by the compatibility conditions and by the governing equation [Eq. (4)] evaluated on the body. The compatibility condition is obtained by differentiating φ along the body

$$(\partial\varphi/\partial r) = (\partial\varphi/\partial x)(dx/d\tau) + (\partial\varphi/\partial r)(dr/d\tau) \text{ on } B$$

or

$$1 = p_B + q_B \varepsilon \bar{R}'(\tau) \quad (7)$$

The other equation is obtained by affixing the suffix B to each term in Eq. (4)

$$(u_B p_B + v_B q_B)^2 - c_B^2(p_B^2 + q_B^2) = 0 \quad (8)$$

So far these manipulations do not solve anything since the velocity components are still unknown. However, this representation for the bicharacteristics can be used to generate a successive approximation scheme whereby the bicharacteristic lines can be found to the same order of accuracy as the velocity components are known. For sonic boom calculations, first-order approximations are usually sufficient. To set up the interactive system of equations, let a perturbation series be postulated as

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 \qquad q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2$$
$$r = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \qquad u = u_0 \qquad + \varepsilon^2 u_2$$
$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 \qquad v = \qquad + \varepsilon^2 v_2 \quad (9)$$

The suffix zero in the series represents conditions in the undisturbed stream. The velocity perturbations are of second order since they vary as the cross-sectional area of the slender body. The local sound speed can be found from Bernoulli's Equation as

$$c^2 = c_0^2 + \varepsilon^2 k u_2 + 0(\varepsilon^4)$$

where $k = -(\gamma - 1)u_0$.

If Eq. (9) is substituted into Eqs. (6–8), the boundary conditions are expressed to order ε^2 as

$$0 = (x_{0B} - \tau) + \varepsilon x_{1B} + \varepsilon^2 x_{2B}$$
$$0 = r_{0B} + \varepsilon[r_{1B} - \bar{R}(\tau)] + \varepsilon^2 r_{2B} \quad (6')$$
$$0 = p_{0B} - 1 + \varepsilon[q_{0B}\bar{R}'(\tau) + p_{1B}] + \varepsilon^2[q_{1B}\bar{R}'(\tau) + p_{2B}] \quad (7')$$
$$\begin{aligned} 0 = u_0^2 p_{0B}^2 - c_0^2 p_{0B}^2 - c_0^2 q_{0B}^2 + \varepsilon(2u_0^2 p_{0B} p_{1B} \\ - 2c_0^2 p_{0B} p_{1B} - 2c_0^2 q_{0B} q_{1B}) \\ + \varepsilon^2[u_0^2 p_{1B}^2 - c_0^2 p_{1B}^2 - c_0^2 q_{1B}^2 \\ + 2u_0^2 p_{0B} p_{2B} + 2u_0 u_{2B} p_{0B}^2 \\ + 2u_0 v_{2B} p_{0B} q_{0B} - 2c_0^2 p_{0B} p_{2B} \\ - 2c_0^2 q_{0B} q_{2B} - k u_{2B}(p_{0B}^2 + q_{0B}^2)] \end{aligned} \quad (8')$$

Since each power of ε in each of Eqs. (6′–8′) must vanish, the boundary conditions are obtained as

$$x_{0B} = \tau, \quad x_{1B} = 0, \quad x_{2B} = 0$$
$$r_{0B} = 0, \quad r_{1B} = \bar{R}(\tau), \quad r_{2B} = 0$$
$$p_{0B} = 1, \quad p_{1B} = \beta\bar{R}'(\tau), \quad p_{2B} = [\beta\bar{R}'(\tau)]^2$$
$$q_{0B} = -\beta, \quad q_{1B} = -\beta^2\bar{R}'(\tau)$$
$$q_{2\beta} = -\beta^3\bar{R}'(\tau)^2 + (\beta u_0 v_{2B} - u_0 u_{2B} + \tfrac{1}{2}k u_{2B} M^2)/\beta c_0^2 N \quad (10)$$

Now that the appropriate boundary conditions have been formulated, the system of differential equations given by Eq. (5) must be analyzed. Without any approximation, the right side of $d\varphi/dt$ vanishes since H is homogeneous of degree 2 in p and q. Thus φ is constant along the bicharacteristic, and by Eq. (6a), this constant is equal to τ. Henceforth, the quantity τ which arose as a parameter describing the boundary manifold may be taken as the characteristic parameter.

The remaining four equations are simplified in exactly the same manner as were the boundary conditions. Each of the equations is expanded in a perturbation series up to $0(\varepsilon^2)$, and the coefficient of ε in each is made to vanish. These 12 equations are

$$\partial x_0/\partial t = 2\lambda\beta^2 c_0^2 p_0, \quad \partial x_1/\partial t = 2\lambda\beta^2 c_0^2 p_1$$
$$\partial r_0/\partial t = -2\lambda c_0^2 q_0, \quad \partial r_1/\partial t = -2\lambda c_0^2 q_1$$
$$\partial p_0/\partial t = 0, \quad \partial p_1/\partial t = 0$$
$$\partial q_0/\partial t = 0, \quad \partial q_1/\partial t = 0$$
$$\partial x_2/\partial t = 2\lambda(\beta^2 c_0^2 p_2 + 2u_0 u_2 p_0 - k u_2 p_0 + u_0 v_2 q_0)$$

$$\partial r_2/\partial t = 2\lambda(u_0 v_2 p_0 - c_0^2 q_2 - k u_2 q_0)$$

$$\frac{\partial p_2}{\partial t} = -2\lambda\left[u_0 p_0\left(p_0\frac{\partial u_2}{\partial x_0} + q_0\frac{\partial v_2}{\partial x_0}\right) - k\frac{\partial u_2}{\partial x_0}(p_0^2 + q_0^2)\right]$$

$$\frac{\partial q_2}{\partial t} = -2\lambda\left[u_0 p_0\left(p_0\frac{\partial u_2}{\partial r_0} + q_0\frac{\partial v_2}{\partial r_0}\right) - k\frac{\partial u_2}{\partial r_0}(p_0^2 + q_0^2)\right] \quad (11)$$

The partial derivative notation just used indicates that derivatives are to be taken with τ constant. It is useful to let the constant parameter λ be equated to $1/(2c_0^2\beta)$.

These equations can be solved in a recursive manner. The terms with suffix 0 and 1 are independent of the flow perturbations, so they can be solved at once

$$x = (\tau + \beta t) + \varepsilon(\beta^2 t\bar{R}')$$
$$r = t \qquad + \varepsilon(\bar{R} + \beta t\bar{R}') \quad (12)$$

The terms with suffix 2 are more complicated since they involve the perturbation velocities. However, both u and v can be calculated to $O(\varepsilon^2)$ by using the usual linear theory. According to the supersonic area rule,[8] the linearized perturbations in the region $(x - \beta r)/r \ll 1$ can be expressed in terms of Whitham's F-function as

$$u_2 = -u_0/(2\beta r)^{1/2} F(x - \beta r)$$
$$v_2 = \beta u_0/(2\beta r)^{1/2} F(x - \beta r) \quad (13)$$

where

$$F(x - \beta r) = \frac{1}{2\pi}\int_0^{x-\beta r} \frac{\partial^2}{\partial \xi^2}[\pi \bar{R}(\xi)]^2 \frac{d\xi}{[\xi - (x - \beta r)]^{1/2}} \quad (14)$$

for a smooth, slender body. If x and r are replaced by their first-order counterparts [Eq. (12)], the perturbations are rewritten as

$$u_2 = [-u_0/(2\beta)^{1/2}]\,[F(S)/(T)^{1/2}]$$
$$v_2 = -\beta u_2, \quad S = r - \beta\varepsilon\bar{R}, \quad T = t + \beta\varepsilon\bar{R}'t + \varepsilon\bar{R} \quad (15)$$

This expression can be substituted into the equations for the direction number perturbations p_2 and q_2. Once p_2 and q_2 are obtained, they are, in turn, substituted into the expression for x_2 and r_2 given in Eq. (11). After applying the boundary conditions and simplifying the representations for x_2 and r_2 by retaining only terms of $O(\varepsilon^2)$, the final expressions for the bicharacteristic are

$$x = (\beta t + \tau) + \varepsilon(\beta^2 t\bar{R}') + \varepsilon^2\Big\{\beta^3 t\bar{R}'^2$$
$$- [(\gamma + 1)M^2/\beta]\,[F/(2\beta)^{1/2}]\,[2/(1 + \beta\varepsilon\bar{R}')]\,[(T)^{1/2} - (\varepsilon\bar{R})^{1/2}]$$
$$- M^2\beta\frac{F}{(2\beta)^{1/2}}\frac{2}{1 + \beta\varepsilon\bar{R}'}[(T)^{1/2} - (\varepsilon\bar{R})^{1/2}]\Big\}$$

$$r = t + \varepsilon(\bar{R} + \beta t\bar{R}') + \varepsilon^2\Big\{\beta^2 t\bar{R}'^2$$
$$- \frac{(\gamma + 1)M^4}{2\beta^2}\frac{F}{(2\beta)^{1/2}}\frac{2}{1 + \beta\varepsilon\bar{R}'}[(T)^{1/2} - (\varepsilon\bar{R})^{1/2}]$$
$$+ \gamma M^2\frac{F}{(2\beta)^{1/2}}\frac{2}{1 + \beta\varepsilon\bar{R}'}[(T)^{1/2} - (\varepsilon\bar{R})^{1/2}]\Big\} \quad (16)$$

This representation gives the values of x and r on the bicharacteristics $\tau = \text{const}$ in terms of the distance t along the curve for a slender, axisymmetric body.

At this point it is instructive to point out the differences between this approach and other methods currently in use. The usual Whitham procedure would stretch only the x (streamwise variable), leaving r unstretched. The corrected characteristic would be[4]

$$x = \beta r - [(\gamma + 1)M^4/\beta]\,[\varepsilon^2 F(\tau)/(2\beta)^{1/2}]\,(r)^{1/2} + \tau \quad (17)$$

But, using Eq. (16) to form the expression for $x - \beta r$,

$$x - \beta r = \varepsilon\beta\bar{R} - [(\gamma + 1)M^4/\beta]\,[\varepsilon^2 F(S)/(2\beta)^{1/2}]$$
$$[(T)^{1/2} - (\varepsilon\bar{R})^{1/2}]/(1 + \beta\varepsilon\bar{R}') + \tau \quad (18)$$

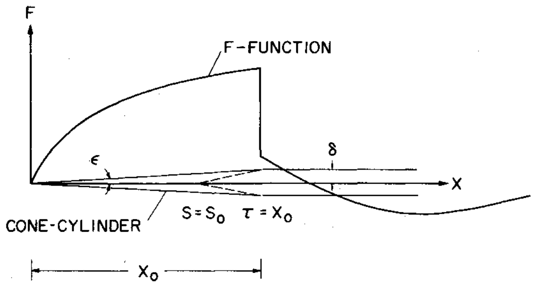

Fig. 1 *F*-function for a cone-cylinder.

If $\beta\varepsilon\bar{R}'$ and $\beta\varepsilon\bar{R}$ are very small, Eq. (15) shows $S \sim \tau$ and $T \sim t \sim r$. Furthermore, if $r \gg \varepsilon\bar{R}$ Eq. (18) reduces to Eq. (17). Equation (16) is better than Eq. (17) in those instances where $\beta\varepsilon\bar{R}$ is not vanishingly small (i.e., at higher Mach numbers). Some recent progress in developing methods for streamwise coordinate stretching has indicated how these high Mach number effects could be treated in special cases.[1,9]

Oswatitsch[10] has developed a theory in which all of the independent variables are stretched. But the parameters into which the independent variables are expressed are related to the exact characteristics by choosing one parameter for each characteristic family. This procedure contrasts with the present method in that in this paper all of the parameters are related to the downstream characteristic family alone. Oswatitsch's method must be used when more than one family of characteristics is important (i.e., reflection problems, third-order effects involving characteristics reflecting off shocks, etc.). In the problem treated here the family of characteristics that is most important as far as shock waves are concerned is expressed in terms of the parameters natural to this family alone.

The representation given in Eq. (16) is generally multivalued for decelerating flows, so shock waves must be inserted to cut off the characteristics before they intersect. The procedure for the shock calculation is given in Sec. III.

III. Parametric Representation of the Shock Wave

A direct way to determine the shock wave is to start from the physical ideas represented by Eq. (4). If the shock speed is given by W, and $\psi(x, r) = 0$ is the equation for the shock, the condition that the magnitude of the velocity normal to $\psi = 0$ is equal to W can be written as

$$(u\,\partial\psi/\partial x + v\,\partial\psi/\partial r)^2 - W^2[(\partial\psi/\partial x)^2 + (\partial\psi/\partial r)^2] = 0 \quad (19)$$

Across the shock wave the flow quantities change abruptly. If suffix 0 indicates upstream conditions and suffix 2 indicates conditions downstream of the shock, Eq. (19) holds with either suffix 0 or 2 attached to u, v, and W. For the bow shock, Eq. (19) will be analyzed for the case when suffix 0 is used. This choice gives the simplest representation since both u_0 and v_0 are constant. W_0 may be found by applying the conservation laws across an element of shock. The formula for W_0 is[6]

$$W_0^2 = (\rho_2/\rho_0)(P_2 - P_0)/(\rho_2 - \rho_0) \quad (20)$$

Equation (20) can be simplified for weak shocks. Assuming an adiabatic equation of state relating p_2 and ρ_2, the small perturbation form of Eq. (20) is

$$W_0^2 = c_0^2 - [(\gamma + 1)M^2/2]\,(\varepsilon^2 u_2/u_0) \quad (21)$$

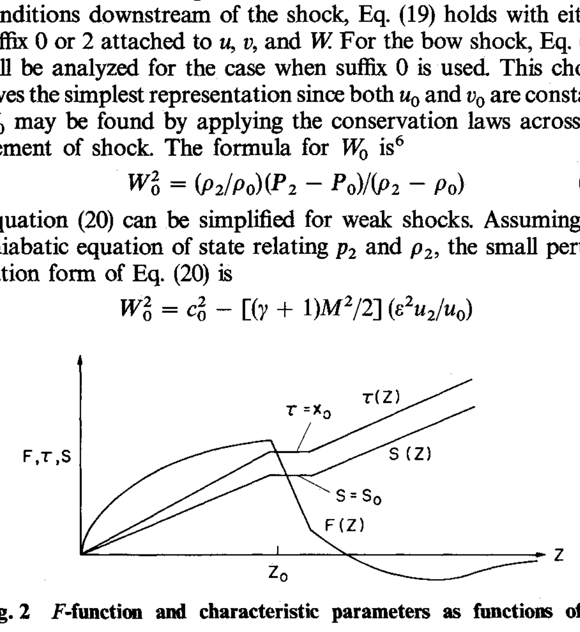

Fig. 2 *F*-function and characteristic parameters as functions of *Z*.

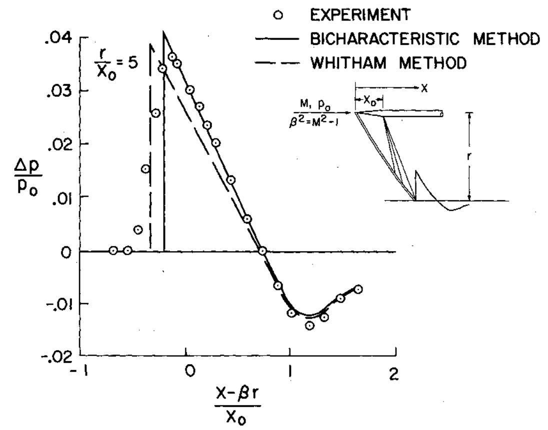

Fig. 3 Cone-cylinder shock wave signature; $M = 2.96$

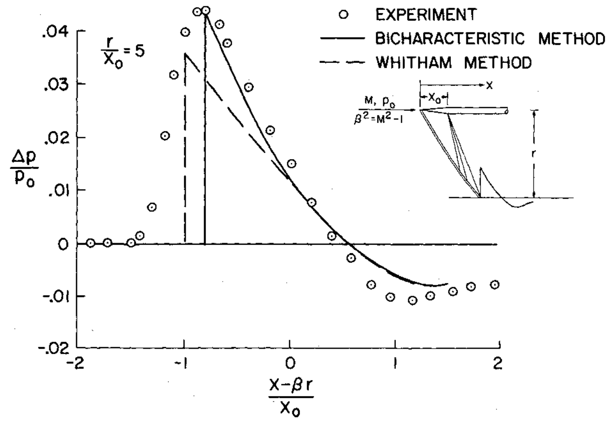

Fig. 5 Cone-cylinder shock wave signatures; $M = 4.63$.

where u_2 is the perturbation velocity downstream of the shock as given in Eq. (15).

Let a parametric representation of the shock wave $\psi(x, r) = 0$ be

$$x = \beta t_0 + \tau_0(t_0)$$

$$r = t_0 \tag{22}$$

The problem is to find the functional relation $\tau_0 = \tau_0(t_0)$ that satisfies Eq. (19). The x and r derivatives of ψ can be expressed in terms of τ_0 as

$$\partial\psi/\partial x = 1, \quad \partial\psi/\partial r = -(d\tau_0/dt_0) - \beta \tag{23}$$

If $u = u_0$, $v = 0$, and the direction numbers given by Eq. (23) are substituted into Eq. (19), a quadratic equation for $d\tau_0/dt_0$ is obtained. Expansion to second order yields

$$d\tau_0/dt_0 = [(\gamma + 1)/4]\,(M^4/\beta)\,(\varepsilon^2 u_2/u_0)$$

But u_2 is a function of S_2 and T_2, which is related to τ_0 and t_0 through Eqs. (16) and (22). The final set of equations that are sufficient to calculate $\tau_0 = \tau_0(t_0)$ are

$$d\tau_0/dt_0 = -\tfrac{1}{2}[(\gamma + 1)M^4/(2\beta)^{3/2}]\,F(S_2)/(T_2)^{1/2}$$

$$\tau_0 = S_2 - \frac{(\gamma + 1)M^4}{(2\beta)^{3/2}} F(S_2) \frac{2}{1 + \varepsilon\beta\bar{R}'}\,[(T_2)^{1/2} - (\varepsilon\bar{R})^{1/2}]$$

$$t_0 = T_2 + \beta^2\varepsilon^2\bar{R}'^2 t_2 - \left[\frac{(\gamma + 1)M^4}{2\beta^2} - \gamma M^2\right]\frac{F(S_2)}{(2\beta)^{1/2}}$$

$$\frac{2}{1 + \varepsilon\beta\bar{R}'}\,[(T_2)^{1/2} - (\varepsilon\bar{R})^{1/2}]$$

$$S_2 = \tau_2 - \beta\varepsilon\bar{R}(\tau_2)$$

$$T_2 = t_2 + \beta\varepsilon\bar{R}'(\tau_2)t_2 + \varepsilon\bar{R}(\tau_2) \tag{24}$$

If S_2 and T_2 could be expressed in terms of τ_0 and t_0, and the result substituted into the first of Eqs. (24), a first-order differential equation could be obtained. In practice, these operations are impossible so a numerical approach was used.

Once the shock wave is obtained, the pressure rise across the shock is calculated from

$$\Delta P/P_0 = -[\gamma M^2\varepsilon^2 u_2(T_{2_{sh}}, S_{2_{sh}})/u_0] \tag{25}$$

The entire signature is then obtained by incrementing $S_{2_{sh}}$, calculating T_2 by solving the bicharacteristic equation at the chosen value of r, and using the S_2 and T_2 in Eq. (25).

IV. Application to the Cone-Cylinder

Consider a cone-cylinder with a radius δ at the shoulder $x = X_0$ (see Fig. 1). Using the notation introduced previously, the radius distribution is given by

$$R(x) = \begin{cases} (\delta/X_0)x & x < X_0 \\ \delta & x > X_0 \end{cases} \tag{26}$$

The small parameter ε is the tangent of the cone semiangle δ/X_0. The F-function for this configuration is not simply the quantity given by Eq. (14) but includes a second term due to the slope discontinuity at the shoulder. Following Lighthill,[11] the F-function is written as

$$F(S) = 2(S)^{1/2} - 2(S - S_0)^{1/2}H(S - S_0) - 2S_0/(2\beta\varepsilon X_0)^{1/2}$$

$$h[(S - X_0)/\beta\varepsilon X_0]$$

$$S = \tau - \beta\varepsilon\tau, \quad S_0 = X_0 - \beta\varepsilon X_0 \tag{27}$$

where $H(x)$ is the Heaviside step function and $h(x)$ is Lighthill's h-function.[11]

For a given value of the characteristic parameter τ, F can be calculated through the parameter S. Note that the parameter S could also be used as the independent parameter. The F-function is sketched in Fig. 1.

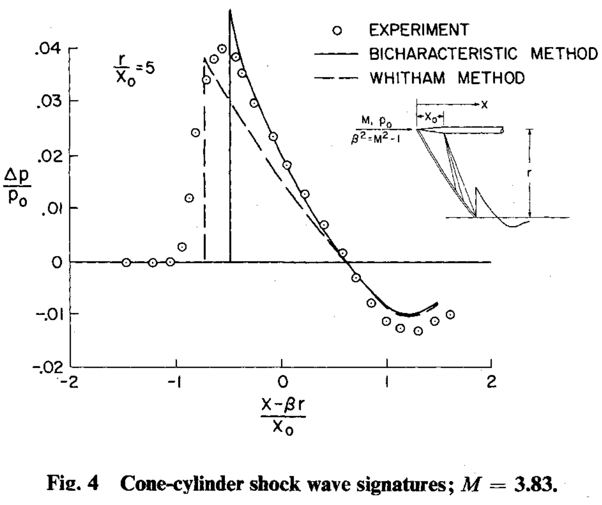

Fig. 4 Cone-cylinder shock wave signatures; $M = 3.83$.

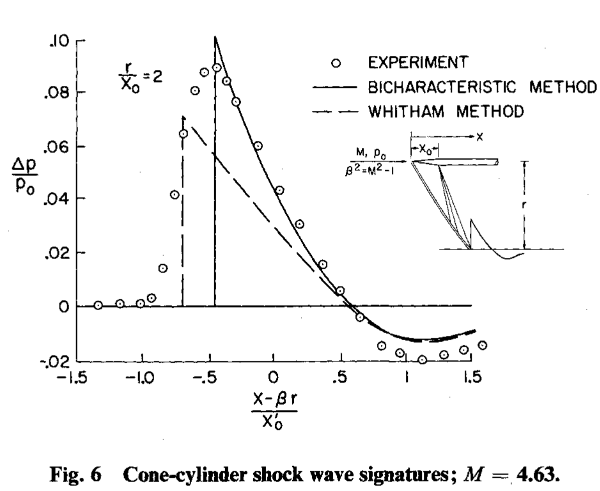

Fig. 6 Cone-cylinder shock wave signatures; $M = 4.63$.

The representation given by Eq. (27) is awkward if the parameter τ (or S) is used because the F-function can take on a whole range of values for the value $\tau = X_0$ (or $S = S_0$). To facilitate the following calculations, a new parameter Z is introduced. This parameter, which could be imagined as an arc length along the F-function, serves to relate each point on the F-function to a single characteristic. In this context, consider Z to be an independent parameter such that each of the variables F, S, and τ are single valued functions of this parameter. Figure 2 shows how the various parameters are represented in terms of Z. This artifice is peculiar only to the cone-cylinder and is not related to the bicharacteristic method.

The entire system of Eqs. (24) were solved numerically for $\tau_0(t_0)$. Lighthill's h-function was approximated by a quartic, and the differential Eq. (24) was solved with the aid of a Runge-Kutta finite-difference method.

Results are presented in Figs. 3–6 for comparison with a series of experimentally determined signatures. These experiments were conducted at Langley Research Center[12] on a slender cone-cylinder. The cone-cylinder has a 6.44° semiangle nose and a shoulder 5.08 cm (2 in.) from the vertex. The radius of the cylindrical portion is 0.573 cm (0.226 in.). Signatures are shown at approach Mach numbers of 2.96, 3.83 and 4.63 at a miss distance of 25.4 cm (10 in.) (Figs. 3, 4, and 5) and at a Mach number of 4.63 at a miss distance of 10.15 cm (4 in.) (Fig. 6). Also presented for comparison are results based on the Whitham theory [i.e., based on Eq. (17)] with the Lighthill correction for the shoulder.§ These figures show that significantly better agreement with experiment is obtained, especially at the higher Mach numbers, when the bicharacteristic theory is used. One important criteria with which to judge the theory is its accuracy in predicting the slope of the expansion portion of the shock wave. This effect is well predicted by the bicharacteristic method, while a systematic underprediction is evident with the Whitham theory. It is probable that the spreading effect in the experimental signatures is due to either pressure differentials across the probe orifice or to probe and model vibrations.

V. Conclusions

It has been shown by calculation and comparison with experiment that a major reason for the discrepancy between calculated sonic boom signatures based on the usual Whitham theory and measured signatures can be traced to an imprecise location of the first order characteristics. It is expected that this effect is most important in the midfield regions. This must be so because near-field signatures are sensitive to the approximation used for the velocity components (i.e., non-F-function behavior), while the far-field pattern must be essentially correct as given by the Whitham theory. (Notice that the signatures given in Figs. 3–6 tend to agree near $\Delta P/P_0 = 0$. This portion of the signature is most important in the far-field.)

§ For the conditions adopted in this paper, the computational algorithm developed by Hayes et al.[13] would yield a signature similar to the one calculated from the Whitham theory.

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